Dynamics and dimension

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Recall the regularity conjecture for simple nuclear C^*-algebras:

CONJECTURE (Toms–W, 2007/08)
For a nuclear, separable, simple, unital and nonelementary C^*-algebra A, t.f.a.e.:

(i) A has finite nuclear dimension
(ii) A is $\mathcal{Z}$-stable (i.e., $A \cong A \otimes \mathcal{Z}$)
(iii) A has strict comparison of positive elements.

Let us confirm the conjecture for a concrete class of C^*-algebras.

THEOREM
The conjecture above holds for the class

$$\begin{aligned}\mathcal{E} \;=\; & \{\mathcal{C}(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ compact, metrizable, infinite,} \\ & \alpha \text{ induced by a uniquely ergodic, minimal homeomorphism}\}.\end{aligned}$$

Moreover, the regularity properties ensure classification by ordered K-theory in this case.

We outline parts of the proof, mainly to indicate that the conjecture whitnesses much of the complexity of Elliott's conjecture.

PROOF

(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) hold in full generality; shown by W and Rørdam, respectively.

(ii) $\Longrightarrow$ (i): Let $A := \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ be as in the theorem. For $x \in X$, let

$$A_x := \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\})) \subset A.$$

$A_x \otimes M_{p^\infty}$ is monotracial, $\mathcal{Z}$-stable and ASH, hence TAF (W, 2006).
$A \otimes M_{p^\infty}$ is TAF as well (Strung–W, using ideas of Lin–Phillips, 2009).
$A \otimes \mathcal{Z}$ is classifiable, hence isomorphic to a model with finite nuclear dimension (Lin, Lin–Niu, Elliott, Villadsen, W,...).

(iii) $\Longrightarrow$ (ii): Let A and A_x be as above. If A has strict comparison, then so has A_x (Phillips).
A_x is ASH, monotracial, with strict comparison, hence is $\mathcal{Z}$-stable (W).
As before, $A_x \otimes M_{p^\infty}$ is TAF, is classifiable, has finite nuclear dimension.
Take another point $y \in X$ (not in the same orbit as x), then A_y has finite nuclear dimension as well.
But then A has finite nuclear dimension (Toms–W), hence is $\mathcal{Z}$-stable. ∎

(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) do not involve the crossed product structure;
(iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) do.

THEOREM (Toms–W, Lin, Lin–Niu, W, Strung–W,...)
If $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is as above and, aditionally, $\dim X < \infty$, then $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ satisfies the regularity conditions.
Moreover, in this case we have classification by ordered K-theory.

QUESTIONS

- Can we confirm the conjecture for more general classes of crossed products?

  What about more general group actions, or arbitrary trace spaces?
- Can we observe similar phenomena at the level of the underlying dynamical systems themselves?

We will make a first attempt by introducing dynamical versions of topological dimension.

DEFINITION (Hirshberg–W–Zacharias)
Let A be a unital C^*-algebra and α an automorphism of A.
We say (A, α) has Rokhlin dimension at most n, $\dim_{\mathrm{Rok}}(A, \alpha) \leq n$, if the following holds:

For every finite subset $\mathcal{F} \subset A$, $L \in \mathbb{N}$ and $\epsilon > 0$ there are c.p.c. order zero maps

$$\varphi^{(i)} : \mathbb{C}^L \oplus \mathbb{C}^{L+1} \to A, \; i = 0, \ldots, n,$$

such that

- $\|\alpha(\varphi^{(i)}(e_j^{(L)})) - \varphi^{(i)}(e_{j+1}^{(L)})\| < \epsilon$ and
  $\|\alpha(\varphi^{(i)}(e_j^{(L+1)})) - \varphi^{(i)}(e_{j+1}^{(L+1)})\| < \epsilon$ for all j (cyclic)
- $\|[\varphi^{(i)}(x), a]\| < \epsilon$ for all $a \in \mathcal{F}$, $i = 0, \ldots, n$, and for all normalized $x \in \mathbb{C}^L \oplus \mathbb{C}^{L+1}$
- $\sum_{i=0}^{n} \varphi^{(i)}(\mathbf{1}_{\mathbb{C}^L} \oplus \mathbf{1}_{\mathbb{C}^{L+1}}) \geq \mathbf{1}_A$.

REMARKS

- We think of $n+1$ as the number of colors, and of L and $L+1$ as the length of the towers.
- $\dim_{\mathrm{Rok}}(A, \alpha) = 0$ is equivalent to α having one of the "classical" Rokhlin properties as considered by Kishimoto et alii.
- Similar as for the Rokhlin properties, one might consider some obvious modifications of this definition (e.g. vary the tower lengths; consider non-cyclic vs. cyclic shift...).

  These are not all equal, but having finite Rokhlin dimension in our sense is usually equivalent to having finite Rokhlin dimension in the modified sense.
- If (A, α) comes from a classical dynamical system (X, T), then we may rephrase this in terms of open coverings rather than partitions of unity.

  At the same time, one may in addition ask the open coverings to become finer and finer.

DEFINITION Let X be compact, metrizable, infinite, and T a minimal homeomorphism of X. We say (X, T) has dynamic dimension at most n, $\dim(X, T) \leq n$, if the following holds:
For any open cover $\mathcal{U}$ of X and any $L \in \mathbb{N}$, there is a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})$$

of open subsets such that

- $T(U_{k,l}^{(i)}) = U_{k,l+1}^{(i)}$ for $i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L-1\}$
- for each fixed $i \in \{0, \ldots, n\}$ the sets $U_{k,l}^{(i)}$ are pairwise disjoint
- $(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})$ is an open cover of X refining $\mathcal{U}$.

REMARK We think of $n + 1$ as the number of colors, of $K^{(i)}$ as the the number of towers of color i, and of L as the length of the towers.

THEOREM (Hirshberg–W–Zacharias, 2011)
Let X be compact, metrizable, infinite, and $\alpha \in \mathrm{Aut}\,(\mathcal{C}(X))$ induced by a minimal homeomorphism T. Suppose X is finite dimensional.

Then, $(\mathcal{C}(X), \alpha)$ has finite Rokhlin dimension and (X, T) has finite dynamic dimension. In fact, we have

$$\dim_{\mathrm{Rok}}(\mathcal{C}(X), \alpha) \leq 2(\dim X + 1) - 1$$

and

$$\dim(X, T) \leq 2(\dim X + 1)^2 - 1.$$

While the statement only involves the dynamical system itself, our current proof makes heavy use of the fine structure of the crossed product C^*-algebra.

It is based on carefully analyzing the representation theory of algebras of the form

$$A_Y := \mathrm{C}^*(\mathcal{C}(X), u\,\mathcal{C}_0(X \setminus Y)) \subset \mathcal{C}(X) \rtimes_\alpha \mathbb{Z},$$

where $Y \subset X$ is closed with nonempty interior.

DEFINITION

We say a c.p.c. map

$$\varphi : F = M_{r_1} \oplus \ldots \oplus M_{r_s} \to A_Y$$

is η-compatible, if there is a c.p.c. map

$$\varphi^\sharp : D \to \mathcal{C}(X) \subset A_Y$$

such that

(i) $\|\varphi(e) - \varphi^\sharp(e)\| \leq \eta\|e\|$ for $e \in D_+$

(ii) $\varphi^\sharp(D^\sigma) \subset \mathcal{C}_0(X \setminus Y)$

(iii) $\|u\varphi^\sharp(e) - \varphi(Se)\| \leq \eta\|e\|$ for $e \in D_+^\sigma$,

where $D^\sigma \subset D \subset F$ denote the (shiftable) diagonal elements and $S = \oplus S_i \in F$ implements the (truncated) shift on D.

THEOREM

Suppose (X, T) is as above, with $\dim X = n < \infty$. Let $Y \subset X$ be closed with nonempty interior.
Then, for any $\eta > 0$ there is a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ of n-decomposable c.p.c. approximations for A_Y with each φ_λ η-compatible.

To prove this, use the standard recursive subhomogeneous decomposition for A_Y.

To use this, approximate $A_{\{y\}}$ by algebras of the form A_Y, and do the same for $A_{\{x\}}$, where $x = T^K y$ for some large enough K.
This will yield finite Rokhlin dimension; finite dynamic dimension is an easy consequence.

DEFINITION

We say (X, T) has slow dynamic dimension growth, if the following holds:
For every open covering $\mathcal{U}$ of X and every $\epsilon > 0$ there are numbers $n, L \in \mathbb{N}$ and a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\},\, k \in \{1, \ldots, K^{(i)}\},\, l \in \{1, \ldots, L\})$$

of open subsets as in the definition of dynamic dimension and, moreover,

$$\frac{n+1}{L} < \epsilon.$$

DEFINITION (Lindenstrauss–Weiss)
(X, T) has mean dimension zero, if the following holds:
For every open covering $\mathcal{U}$ of X and every $\epsilon > 0$ there are a number $L \in \mathbb{N}$ and an open covering $\mathcal{V}$ of X such that

$$\mathcal{V} \text{ refines } \mathcal{U} \vee T^{-1}\mathcal{U} \vee \ldots \vee T^{-L+1}\mathcal{U}$$

and, moreover,

$$\frac{\text{coloring}\#(\mathcal{V})}{L} < \epsilon.$$

REMARK
If (X, T) is uniquely ergodic, then it has mean dimension zero.

THEOREM

If (X, T) has slow dynamic dimension growth, then it has mean dimension zero.
If (X, T) has mean dimension zero and finite Rokhlin dimension, then it has slow dynamic dimension growth.

COROLLARY

If (X, T) is uniquely ergodic and has a finite dimensional, minimal factor, then it has slow dynamic dimension growth.

QUESTIONS

- Does slow dynamic dimension growth (or a variant thereof) imply $\mathcal{Z}$-stability of the crossed product C^*-algebra?
- Could slow dynamic dimension growth (or a variant thereof) serve as a dynamic analogue of $\mathcal{Z}$-stability?

We close with some applications to C^*-dynamical systems.

PROPOSITION (Hirshberg–W–Zacharias, 2010)
Let A be separable and unital, with finite nuclear dimension.
If $\alpha \in \mathrm{Aut}\,(A)$ has finite Rokhlin dimension, then $A \rtimes_\alpha \mathbb{Z}$ has finite nuclear dimension.

THEOREM (Hirshberg–W–Zacharias, 2010)
Let A be separable, unital and $\mathcal{Z}$-stable.
Then, there is a dense G_δ set $\Gamma \subset \mathrm{Aut}\,(A)$ such that each $\alpha \in \Gamma$ has finite Rokhlin dimension.

THEOREM (Hirshberg–W–Zacharias, 2010)
Let A be separable, simple and unital, with finite nuclear dimension.
Then, there is a dense G_δ set $\Gamma \subset \mathrm{Aut}\,(A)$ such that for each $\alpha \in \Gamma$, $A \rtimes_\alpha \mathbb{Z}$ is simple, $\mathcal{Z}$-stable, with finite nuclear dimension.

(Generalized by Matui–Sato in the unique trace case.)

PROOF (Idea)

Let $\mathcal{F}_l \subset A$, $l \in \mathbb{N}$, be an increasing sequence of finite subsets with dense union.

Define open subsets of $\mathrm{Aut}(A)$

$$V_{k,l} := \{\alpha \in \mathrm{Aut}(A) \mid \text{ the Rokhlin condition holds for } k, \mathcal{F}_l \text{ and } 1/l\}.$$

Each $\alpha \in \bigcap_{k,l} V_{k,l}$ will have Rokhlin dimension at most n.

If we can show that each $V_{k,l}$ is dense in $\mathrm{Aut}(A)$, then the Baire category theorem will imply that $\bigcap_{k,l} V_{k,l}$ is also dense in $\mathrm{Aut}(A)$ (the latter is completely metrizable).

With $\alpha \in \mathrm{Aut}(A)$ given, find $\sigma_{k,l} \in \mathrm{Aut}(\mathcal{Z})$ such that $\alpha \otimes \sigma_{k,l} \in \mathrm{Aut}(A \otimes \mathcal{Z})$ satisfies the Rokhlin condition for k, $\mathcal{F}_l \otimes \mathbf{1}_{\mathcal{Z}}$ and $1/l$ (in place of L, $\mathcal{F}$ and ϵ).

Find isomorphisms $\varrho_m : A \stackrel{\cong}{\longrightarrow} A \otimes \mathcal{Z}$ such that

$$V_{k,l} \ni \varrho_m^{-1} \circ (\alpha \otimes \sigma_{k,l}) \circ \varrho_m \stackrel{m\to\infty}{\longrightarrow} \alpha \text{ in } \mathrm{Aut}(A).$$